Ordered phases of XXZ-symmetric spin-1/2 zigzag ladder

M. Zarea[1], M. Fabrizio[1,a], and A.A. Nersesyan[2,3]

[1] International School for Advanced Studies, Via Beirut 4, 34014 Trieste, Italy
[2] The Abdus Salam International Centre for Theoretical Physics, P.O. Box 586, 34100 Trieste, Italy
[3] The Andronikashvili Institute of Physics, Tamarashvili 6, 380077 Tbilisi, Georgia

Abstract. Using bosonization approach, we derive an effective low-energy theory for XXZ-symmetric spin-1/2 zigzag ladders and discuss its phase diagram by a variational approach. A spin nematic phase emerges in a wide part of the phase diagram, either critical or massive. Possible crossovers between the spontaneously dimerized and spin nematic phases are discussed, and the topological excitations in all phases identified.

PACS. 75.10.Pq Spin chain models – 75.40.-s Critical-point effects, specific heats, short-range order – 75.30.Gw Magnetic anisotropy – 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.)

1 Introduction

The appearance of unconventional spin-liquid phases in frustrated Heisenberg models around a critical point separating different magnetic ordered phases is a long standing, intriguing issue which has attracted notable theoretical and experimental interest in recent years [1]. From a theoretical point of view, this is quite a challenging problem which calls for a deep reexamination of standard theories. Conventionally, the stability of a spin-ordered phase may be investigated by spin-wave theory or by more sophisticated field-theoretical approaches based on the non-linear σ-model. Spin-wave theory is in principle able to detect instabilities at any wave-vector, although a reliable description would require a systematic $1/S$ expansion, S being the magnitude of the spin. The non-linear σ-model offers a better description of the critical behavior close to an instability point, yet it has a major limitation. Namely, it takes into account only the long-wavelength Goldstone modes of the ordered state under consideration, but fails to describe the excitations at the wave-vector of the competing ordered state. Hence it does not provide any information about the phase which can emerge under increasing the effect of frustration.

One-dimensional spin models might turn useful to attempt an improvement of the field-theoretical approach in view of an extension to higher dimensions. The simplest example of one-dimensional frustrated spin model is the spin-1/2 Heisenberg chain with antiferromagnetic nearest neighbor exchange J' and frustrating next nearest neighbor exchange J (for a recent review and refer-

ences therein, see Ref. [2]). Besides the general interest, this model might also be relevant for realistic materials, such as Cs_2CuCl_4, in which magnetic Cu ions get arranged into a zig-zag fashion [3], although these compounds are likely more quasi-two than quasi-one-dimensional.

In the classical limit, the J-J' spin chain has a Néel long range order for $j = J/J' < 1/4$, with characteristic momentum $q = \pi/a_0$. In the Néel phase time reversal symmetry is preserved if combined with a translation by one lattice spacing a_0. For larger values of j, a spiral ordering at momentum satisfying $\cos(q\,a_0) = -1/4\,j$ is stabilized at the classical level. There parity and time reversal symmetries are separately broken.

Quantum fluctuations modify the classical phase diagram [4]. By Mermin-Wagner's theorem, spin rotational symmetry cannot be broken in one dimension: the Néel long-range ordered phase turns into a quasi-long range ordered one characterized by power-law decaying correlations of the staggered magnetization. The low-energy effective critical theory is the level-1 Wess-Zumino-Novikov-Witten (WZNW) model (free massless bosons with central charge $C = 1$). On the contrary, the spiral order disappears completely in favor of a spontaneously dimerized phase. The transition point is slightly shifted with respect to the classical value, $j_c \simeq 0.241$ [5,6]. Within the WZNW model formalism, the transition is driven by a perturbation which is marginally irrelevant (relevant) at $j < j_c$ ($j > j_c$). At $j = j_c$ a Berezinskii-Kosterlitz-Thouless transition takes place, and an exponentially small spectral gap opens up in the region $j > j_c$ [7]. The system continuously passes to a two-fold degenerate, spontaneously dimerized, massive phase. Upon further increasing j, the gap reaches its maximum at the exactly solvable Majumdar-Ghosh

point [8], $j = 1/2$, after which it slowly decreases [7]. Even though the ground state of the J-J' chain remains dimerized at all $j > j_c$, above the Majumdar-Ghosh point the system reveals signatures of the classical spiral phase: the spin-spin correlations become incommensurate, as it was shown numerically in references [7,9]. Since this occurs far away from the region of applicability of the $SU(2)_1$ WZNW model, there is little scope to improve the field-theoretical description of the gapless phase at $j < j_c$ to account for incommensurate correlations that emerge well above j_c. More promising is to approach this problem from the opposite side, $j \gg 1$. If $J' = 0$, the even and odd sublattices of the spin chain decouple, and the model effectively describes two decoupled Heisenberg chains, one for each sublattice. Classically this corresponds to the case when the spiral wave number is equal to $q = \pi/2a_0$. Switching on a small J' transforms the model to a weakly coupled two-chain zigzag spin ladder, with the interchain coupling giving rise to a marginally relevant perturbation that opens up a gap and brings the system back to the dimerized phase. In addition it should also move the relevant momentum q away from $\pi/2a_0$ towards π/a_0. Therefore incommensuration and the spectral gap are supposed to appear together in this limit, which makes a field-theoretical description more plausible.

Indeed, in the limit $j \gg 1$, a novel, parity-breaking (twist) perturbation was identified in reference [10] as a natural source of the spin incommensurabilities. The twist term has a tendency to support a finite spin current along the chains which would account for the expected shift of the momentum. However, for the $SU(2)$-symmetric zigzag ladder, the situation still remains rather unclear. Apparently, the appearance of a nonzero spin current is not compatible with the requirement of unbroken spin rotational symmetry (see, however, the discussion in Sect. 7). On the other hand, no reliable information about the actual role of the twist operator at the strong-coupling fixed point can be extracted from the Renormalization Group (RG) analysis [10] because of the perturbative nature of this approach. Thus, the structure of the low-energy effective field theory for the $SU(2)$-symmetric $S = 1/2$ zigzag ladder still remains unknown.

The situation changes much to the better in the presence of strong spin anisotropy. Namely, close to the XX limit, a self-consistent, symmetry-preserving mean-field approach shows that the twist operator can stabilize a new, spin-nematic (chiral) phase [10]. In this doubly degenerate phase a nonzero spin current polarized along the easy axis flows along the ladder, and the transverse spin-spin correlations are incommensurate and may even decay algebraically within some parameter range, as it is the case at the XX point. This picture is supported by recent numerical simulations [11–14]. In particular, the numerical work by Hikihara et al. [11,12] has indeed confirmed the existence of the critical spin-nematic phase in a broad region of the phase diagram for spin-anisotropic chains, both for integer and half-integer spins. NMR experiments with compound CaV_2O2_4 [15] having a spin-1 double-chain zigzag magnetic structure have revealed the gapless

nature of the spectrum, which may be an indication to the critical chiral state of this material. On the other hand, numerical simulations have revealed the existence of a new gapped chiral phase in a very narrow region between the dimerized phase and the critical chiral phase, but, within numerical accuracy, only for integer spins. In that gapped phase, the spin current coexists with dimerization; accordingly, the spin-spin correlations are incommensurate but decay exponentially. Recently the phase diagram for general spin S has been studied by bosonization technique [16] and by means of the non-linear σ-model [17], verifying the existence of both critical and gapped chiral phased for integer spin.

In this paper we present a detailed study of the phase diagram of the XXZ-symmetric, frustrated, spin-1/2 chain in the limit $J' \ll J$. Using a variational analysis of the bosonized Hamiltonian we identify possible phases of the model. In addition to the critical spin-nematic phase and to the commensurate spontaneously dimerized one, we find conditions for the existence of a massive spin-nematic region for the $S = 1/2$ case. We also characterize the topological excitations which occur in each region of the phase diagram.

In the following section we introduce the model and discuss the bosonization approach. In Section 3 we demonstrate how the variational approach can be applied to the twistless ladder, in which only the dimerization operator plays a role. Critical spin nematic phase, driven only by the twist operator, is studied in Section 4. The interplay between dimerization and twist operator and other emerging phases is discussed in Section 5. In Section 6 we discuss a ferromagnetic phase which turns out to be dual to the critical spin nematic phase. In Section 7 the RG approach is implemented to study the interplay between different twist operators at the border of these mutually dual phases. The last section contains conclusions.

2 The model and its low-energy limit

We consider a frustrated spin-1/2 Heisenberg chain with $2L$ sites, described by the Hamiltonian

$$H = \sum_{a=x,y,z} \sum_{n=1}^{2L} \left[J'_a S_n^a S_{n+1}^a + J_a S_n^a S_{n+2}^a \right], \qquad (1)$$

where

$$J_x = J_y = J > 0, J_z = J\Delta,$$
$$J'_x = J'_y = J' > 0, J'_z = J'\Delta'. \qquad (2)$$

In what follows, Δ and Δ' will be treated as independent anisotropy parameters. Upon the transformation

$$\mathbf{S}_{2n} \to \mathbf{S}_1(n), \quad \mathbf{S}_{2n+1} \to \mathbf{S}_2(n),$$

the model (1) is mapped onto the zig-zag spin-1/2 ladder Hamiltonian

$$H = \sum_{a=x,y,z} \sum_{n=1}^{L} \sum_{i=1,2} J_a S_i^a(n) S_i^a(n+1)$$
$$+ \sum_{a=x,y,z} \sum_n J_a' \left[S_1^a(n) + S_1^a(n+1) \right] S_2^a(n). \quad (3)$$

Let us discuss some general properties of this model. In the limit $J = 0$, (1) describes a standard Heisenberg antiferromagnetic chain where the spin-spin correlations are modulated with wavevector $q = \pi$. On the contrary, when $J' = 0$, the equivalent model (3) describes two decoupled spin chains. The modulating wavevector in this case is $q = \pi/2$. When both J and J' are finite, we may expect two possible behaviors of the spin structure factor $S(q) = (2L)^{-1} \sum_{n,m} \langle \mathbf{S}_n \cdot \mathbf{S}_m \rangle e^{iq(n-m)}$: either it is peaked at $q = \pi$ and $q = \pi/2$, or it shows a single peak at an incommensurate q_0 which smoothly moves from $q = \pi/2$ at $J \gg J'$ towards $q = \pi$ when $J' \gg J$. Translated into the zig-zag ladder language, the former case implies that, for n large,

$$\langle \mathbf{S}_1(n) \cdot \mathbf{S}_1(0) \rangle = \langle \mathbf{S}_2(n) \cdot \mathbf{S}_2(0) \rangle$$
$$= F_0(|2n|) + F_\pi(|2n|) + (-1)^n F_{\pi/2}(|n|),$$
$$\langle \mathbf{S}_1(n) \cdot \mathbf{S}_2(0) \rangle = F_0'(|2n+1|) - F_\pi'(|2n+1|),$$
$$\langle \mathbf{S}_2(n) \cdot \mathbf{S}_1(0) \rangle = F_0'(|2n-1|) - F_\pi'(|2n-1|),$$

where $F_0(|n|)$, $F_\pi(|n|)$ and $F_{\pi/2}(|n|)$, as well as the primed ones, are smooth real functions describing the contributions of the $q = 0$, $q = \pi$ and $q = \pi/2$ modes, respectively. The difference between the 1-2 and 2-1 spin-spin correlators, as well as the absence of inversion symmetry $n \to -n$, reflect the fact that the zigzag ladder lacks two Z_2 symmetries – the $1 \leftrightarrow 2$ interchange symmetry and site parity P_S (understood as $P_S^{(1)} \otimes P_S^{(2)}$). However, if the model is gapless or possesses a small spectral gap inducing a macroscopically large correlation length, then site-parity is effectively restored at long distances.

If, apart from $q = 0$, the spin structure factor has a peak at an incommensurate wave vector $q_0 \in [\pi/2, \pi]$, then we expect that

$$\langle \mathbf{S}_1(n) \cdot \mathbf{S}_1(0) \rangle = \langle \mathbf{S}_2(n) \cdot \mathbf{S}_2(0) \rangle$$
$$= F_0(|2n|) + F_{q_0}(|2n|) \cos 2n q_0,$$
$$\langle \mathbf{S}_1(n) \cdot \mathbf{S}_2(n) \rangle = F_0'(|2n+1|)$$
$$+ F_{q_0}'(|2n+1|) \cos q_0(2n+1), \quad (4)$$
$$\langle \mathbf{S}_2(0) \cdot \mathbf{S}_1(n) \rangle = F_0'(|2n-1|)$$
$$+ F_{q_0}'(|2n-1|) \cos q_0(2n-1). \quad (5)$$

We notice that the presence of the modulating factors in (4), (5) makes the the breakdown of P_S even more pronounced and, contrary to the commensurate case, this breakdown will survive the continuum limit we are going to adopt. Thus, the two different types of spin correlations — commensurate or incommensurate — can be distinguished within a continuum, low-energy description by

an asymptotic restoration or breakdown of the site-parity symmetry.

As discussed in the Introduction, in this paper we are going to study the model (3), or equivalently (1), in the limit $J \gg J'$ of weakly coupled chains. That allows us to adopt the well-known continuum description of each XXZ chain based on the bosonization approach [18] (see also Ref. [19] for a recent review) and then treat the interchain coupling as a weak perturbation. Bosonization of the XXZ zigzag spin-1/2 ladder has already been discussed in reference [20] where the attention was paid mostly on the estimation of scaling dimensions of various perturbations without a detailed analysis of the phase diagram of the model. Here we review this procedure in more detail paying attention to the structure of the effective continuum model which will prove important for the subsequent nonperturbative analysis of the phase diagram, based on a variational approach.

We start with the Abelian bosonization of a single XXZ spin-1/2 chain. Its universal, low-energy properties in the gapless Luttinger liquid phase ($-1 < \Delta \leq 1$) are adequately described by the Gaussian model for a massless scalar field $\varphi(x) = \varphi_R(x) + \varphi_L(x)$,

$$H_{XXZ} \to \int dx \, \mathcal{H}_G(x),$$
$$\mathcal{H}_G = (v_s/2) \left[Q^{-1} (\partial_x \varphi)^2 + Q (\partial_x \vartheta)^2 \right]. \quad (6)$$

Here $\vartheta(x) = -\varphi_R(x) + \varphi_L(x)$ is the field dual to $\varphi(x)$, $\pi(x) = -\partial_x \vartheta(x)$ being the momentum conjugate to $\varphi(x)$, $v_s \sim J a_0$ is the velocity of the collective spin excitations, and Q is the the Luttinger liquid parameter which determines the compactification radius of the field φ. The dependence $Q = Q(\Delta)$ in the whole range $-1 < \Delta \leq 1$ is known from the Bethe-ansatz solution: $1/Q = 1 - (1/\pi) \arccos \Delta$. Thus Q varies in the range $\infty > Q \geq 1$ when Δ takes values within the interval $-1 < \Delta \leq 1$. In particular, $Q = 2$ at the XX point ($\Delta = 0$), and $Q = 1$ at the SU(2)-symmetric (Heisenberg) point ($\Delta = 1$). Throughout this paper it will be assumed that $\Delta < 1$ (i.e. $Q > 1$). In this case the perturbation to the Gaussian model (6), $\lambda_U \cos \sqrt{8\pi}\varphi$ ($\lambda_U \sim J\Delta$), which in terms of the Jordan-Wigner fermions originates from Umklapp processes, is strongly irrelevant and will be dropped in what follows.

Since for each chain only the uniform and staggered low-energy modes survive the continuum limit, the corresponding spin densities can be parametrized as follows:

$$\mathbf{S}_i(n) \to a_0 \mathbf{S}_i(x), \quad (x = n a_0)$$
$$\mathbf{S}_i(x) = \mathbf{J}_{i,R}(x) + \mathbf{J}_{i,L}(x) + (-1)^n \mathbf{n}_i(x) \quad (i = 1, 2).$$

Here a_0 is the lattice spacing, $\mathbf{J}_{i,R,L}$ are chiral components of the smooth part of the magnetization of the i-th chain, and $\mathbf{n}_i$ is the staggered magnetization. The latter is even under site parity transformation (P_S) and odd under link parity transformation (P_L). A distinctive feature of the zigzag ladder is that it is invariant under mixed parity: $P_S^{(1)} \otimes P_L^{(2)}$ and $P_L^{(1)} \otimes P_S^{(2)}$. By this symmetry, strongly relevant terms, $n_1^a n_2^a$, which determine the

spin-liquid properties of the unfrustrated spin-1/2 ladders [21], are instead forbidden in model (3). As a result, in the low-energy limit, the interchain perturbation, $\mathcal{H}' = \mathcal{H}_{JJ} + \mathcal{H}_{\text{twist}}$, is contributed by the "current-current" interaction [7,22]

$$\mathcal{H}_{JJ} = 2 \sum_{a=x,y,z} g_a J_1^a J_2^a, \qquad (7)$$

and also by "twist" terms [10] allowed by the $P_S^{(1)} \otimes P_L^{(2)}$ symmetry:

$$\mathcal{H}_{\text{twist}} = \frac{1}{2} \sum_{a=x,y,z} g_a a_0 T^a + \frac{1}{2} g_0 a_0 T^0. \qquad (8)$$

Here

$$T^a = n_1^a \partial_x n_2^a - n_2^a \partial_x n_1^a, \quad T^0 = \epsilon_1 \partial_x \epsilon_2 - \epsilon_2 \partial_x \epsilon_1, \quad (9)$$

are chirally asymmetric operators with conformal spin 1, and $\epsilon_i \simeq (-1)^n \mathbf{S}_i(n) \cdot \mathbf{S}_i(n+1)$ represent the continuum limit of the dimerization operators. (Note that for a single chain $\epsilon(x)$ is even under P_L and odd under P_S.) The coupling constants are given by $g_x = g_y \equiv g_\perp = J' a_0$, $g_z \equiv g_\parallel = J' \Delta' a_0$.

The "current-current" and twist perturbations are of different nature. The former are parity [i.e. $P_{S(L)}^{(1)} P_{S(L)}^{(2)}$] symmetric. If acting alone, provided that the interchain exchange is antiferromagnetic, these lead to spontaneous dimerization of the ground state (see Sect. 3), the existence of massive topological excitations (spinons), and the onset of short-ranged *commensurate* interchain spin correlations [7,23]. The twist terms, whose appearance stems from the frustrated nature of interchain interaction, explicitly break parity. However, by the previous discussion, parity can be broken either in a mild way, which is the case when the leading asymptotics of spin-spin correlations are still commensurate, or more profoundly, i.e. explicitly inducing *incommensurations* in the spin correlations. Both patterns of the low-energy behavior of the system will be discussed below.

In model (3), only the vector part of the twist perturbation, $g_a T^a$, emerges in the continuum limit. The scalar part, $g_0 T^0$, although absent in the bare Hamiltonian, is generated in the course of RG flow [10].

For this reason we will assume that such a term is present at the outset, with a bare amplitude g_0.

Let us first bosonize $\mathcal{H}_{JJ}$. In terms of the rescaled fields, $\phi_i = (1/\sqrt{Q}) \varphi_i$ and $\theta_i = \sqrt{Q} \vartheta_i$, the "currents" $J_i^a = J_{i,R}^a + J_{i,L}^a$, are given by [24]

$$J_i^z = \sqrt{\frac{Q}{2\pi}} \partial_x \phi_i, \quad J_i^+ = -\frac{\zeta}{\pi\alpha} e^{i\sqrt{\frac{2\pi}{Q}}\theta_i} \cos \sqrt{2\pi Q}\phi_i, \quad (10)$$

where α is the short-distance cutoff of the bosonic theory, and $\zeta(Q)$ is a nonuniversal (and yet unknown) positive constant approaching the value 1 in the SU(2) limit. Using the definitions (10) and passing to the symmetric and antisymmetric combinations of the fields, $\phi_\pm = (\phi_1 \pm \phi_2)/\sqrt{2}$,

$\theta_\pm = (\theta_1 \pm \theta_2)/\sqrt{2}$, we find that the longitudinal (zz) part of $\mathcal{H}_{JJ}$ adds to the Gaussian part of the model transforming the latter into

$$\mathcal{H}_G \to \sum_{\sigma=\pm} \frac{v_\sigma}{2} \left[R_\sigma (\partial_x \theta_\sigma)^2 + R_\sigma^{-1} (\partial_x \phi_\sigma)^2 \right], \qquad (11)$$

with

$$\frac{1}{R_\pm} = \frac{v_\pm}{v_s} = \sqrt{1 \pm \frac{g_\parallel Q}{\pi v_s}} = 1 \pm \frac{g_\parallel Q}{2\pi v_s} + O(g_\parallel^2). \qquad (12)$$

The exact dependence of $R_\pm$ on the dimensionless parameter $g_\parallel Q/\pi v_s$ is unknown. Therefore we will restrict ourselves to the case $|g_\parallel| Q/\pi v_s \ll 1$ and keep only linear terms in the expansion (12). For a weak interchain interaction ($|g_\parallel|/\pi v_s \ll 1$), this is justified almost for the whole range $|\Delta| < 1$ except for a narrow region $\Delta + 1 \sim (g_\parallel/\pi v_s)^2$ close to the ferromagnetic transition point. The parameters $R_\pm$ then satisfy the relation $R \equiv R_+ = 1/R_-$ which considerably simplifies the perturbative analysis.

Performing an additional rescaling of the fields, $\phi_\pm = \sqrt{R_\pm} \Phi_\pm$, $\theta_\pm = \sqrt{R_\mp} \Theta_\pm$, for the transverse ($xx$, yy) part of $\mathcal{H}_{JJ}$ one finds:

$$\mathcal{H}_{JJ;\perp} = 2g_\perp \sum_{a=x,y} J_1^a J_2^a = \frac{\lambda_\perp}{\pi\alpha} (\mathcal{D} + \mathcal{F}), \qquad (13)$$

$$\mathcal{D} = \cos \sqrt{4\pi K_+} \Phi_+ \cos \sqrt{4\pi K_-} \Theta_-,$$

$$\mathcal{F} = \cos \sqrt{4\pi/K_-} \Phi_- \cos \sqrt{4\pi K_-} \Theta_-, \qquad (14)$$

where $\lambda_\perp = g_\perp \zeta^2/\pi\alpha$, and

$$K_+ = QR, \quad K_- = R/Q. \qquad (15)$$

To bosonize the twist perturbation (8), we use the bosonization formulas for the staggered magnetization of the $S = 1/2$ XXZ chain (see e.g. [19]):

$$n_i^z = -(C_z/\pi\alpha) \sin \sqrt{2\pi Q} \, \phi_i$$
$$n_i^\pm = (C_x/\pi\alpha) \exp(\pm i\sqrt{2\pi/Q} \, \theta_i), \qquad (16)$$

where $C_a(Q)$ ($a = x, z$) are noniniversal parameters (their exact dependence on Q was recently found in Refs. [25,26]). Then, in terms of the fields $\Phi_\pm$, $\Theta_\pm$, the twist term becomes:

$$\mathcal{H}_{\text{twist}} = \sum_{i=1,2,3} \lambda_i \mathcal{O}_i, \qquad (17)$$

with

$$\lambda_1 \sim C_x^2 g_\perp/\alpha, \quad \lambda_{2,3} \sim C_z^2 (\pm g_\parallel + g_0)/\alpha, \qquad (18)$$

and three bosonized twist operators $\mathcal{O}_{1,2,3}$ related to T^i ($i = 0,1,2,3$) as follows:

$$\mathcal{O}_1 = T^x + T^y = \frac{2}{\sqrt{K_+}} \partial_x \Theta_+ \sin \sqrt{4\pi K_-} \Theta_-, \quad (19)$$

$$\mathcal{O}_2 = \frac{T^0 + T^z}{2} = \sqrt{K_+} \partial_x \Phi_+ \sin \sqrt{\frac{4\pi}{K_-}} \Phi_-, \qquad (20)$$

$$\mathcal{O}_3 = \frac{T^0 - T^z}{2} = \frac{1}{\sqrt{K_-}} \partial_x \Phi_- \sin \sqrt{4\pi K_+} \Phi_+. \quad (21)$$

Thus, the bosonized continuum version of our model, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{JJ;\perp} + \mathcal{H}_{\text{twist}}$, represents a Gaussian field theory of two scalar fields,

$$\mathcal{H}_0 = \sum_{\sigma=\pm} \mathcal{H}_0^{(\sigma)} = \sum_{\sigma=\pm} \frac{v_\sigma}{2} \left[(\partial_x \Phi_\sigma)^2 + (\partial_x \Theta_\sigma)^2 \right], \quad (22)$$

with perturbations (13) and (17) which couple the (+) and (−) channels together. Since $\mathcal{H}_0$ is perturbed in a relevant way, the relationship between the coupling constants of the original model (1) and the parameters of $\mathcal{H}$, obtained within our weak-coupling approach, is not to be trusted. For this reason we will consider those parameters as independent. Namely, we will treat the Hamiltonian $\mathcal{H}$ as a low-energy effective theory for a most general class of frustrated zigzag spin-1/2 ladders, sharing the same symmetry properties with the model (1).

The scaling dimensions of the perturbing operators are:

$$d_{\mathcal{D}} = K_+ + K_-, \quad d_{\mathcal{F}} = K_- + \frac{1}{K_-},$$

$$d_1 = 1 + K_-, \qquad d_2 = 1 + \frac{1}{K_-}, \quad d_3 = 1 + K_+. \quad (23)$$

Their relevance ($d < 2$) or irrelevance ($d > 2$) can be understood from Figure 1 where the plane (K_-, K_+) is shown. The point $K_+ = K_- = 1$ corresponds to the SU(2)-symmetric zig-zag ladder where all perturbations (including the Umklapp term) are marginal [10]. This point and its close vicinity will not be discussed in this paper. Due to the condition $K_+/K_- = Q^2 \geq 1$, the physical part of the (K_-, K_+) plane lies above the line $K_+ = K_-$ and can be divided into four sectors in which at least one twist operator is relevant:

$$\begin{aligned}
&\text{sector A:} \quad d_1 < 2, \quad d_{\mathcal{F}} \geq 2, \quad d_2, d_3, d_{\mathcal{D}} > 2, \\
&\text{sector B:} \quad d_1 < d_{\mathcal{D}} < 2, \qquad d_2, d_3, d_{\mathcal{F}} > 2, \\
&\text{sector C:} \quad d_{\mathcal{D}} < d_1 < d_3 < 2, \qquad d_2, d_{\mathcal{F}} > 2, \\
&\text{sector D:} \quad d_2 < 2, \quad d_{\mathcal{F}} \geq 2, \quad d_1, d_3, d_{\mathcal{D}} > 2. \quad (24)
\end{aligned}$$

Notice that, except for the operator $\mathcal{D}$, all other perturbing operators have a nonzero Lorentz spin: $S_{1,2,3} = 1$, $S_{\mathcal{F}} = 2$. Strictly speaking, the conventional criterium of relevance does not apply to such operators (see e.g. Ref. [19]) because in higher orders of perturbation theory they can generate relevant scalar perturbations. Using standard fusion rules for the Gaussian model, we have analyzed the structure of various terms appearing in the second order of perturbation theory. There are marginal terms leading to small corrections to the parameters $K_\pm$ and the velocities $v_\pm$, as well as those which renormalize the already existing coupling constants. Besides, new scalar operators $\cos 2\sqrt{4\pi K_+}\Phi_+$, $\cos 2\sqrt{4\pi/K_-}\Phi_-$, $\cos 3\sqrt{4\pi K_-}\Theta_-$ and $\cos 2\sqrt{4\pi K_-}\Theta_-$ are generated. The first two of them have scaling dimensions $4QR$ and $4Q/R$, respectively, and are therefore strongly irrelevant (since $Q > 1$, $R \sim 1$). The third operator has dimension $9K_- = 9R/Q$ and becomes relevant roughly at $Q > 9/2$,
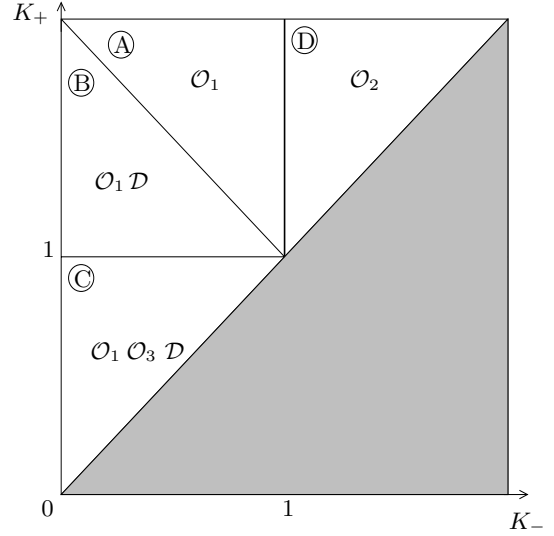

Fig. 1. The phase space of the model parametrized by K_+ and K_- depending on the in-chain and interchain spin anisotropy. The shaded region is unphysical. Sectors A, B, C, D are characterized by sets of relevant operators. $\mathcal{O}_{1,2,3}$ are twist operators while $\mathcal{D}$ is the dimerization operator.

which is a region of the ferromagnetic intrachain exchange, far away from the XX point. This region will not be considered here. Finally, the last perturbation $\sim \cos 2\sqrt{4\pi K_-}\Theta_-$ has dimension $4K_- = 4R/Q$ and thus becomes relevant at $K_- < 1/2$, which corresponds to a vicinity of the XX point $Q = 2$. However, even in that case its role is subdominant, as we will show later.

Thus, the continuum model we will be dealing with in the remainder of this paper reads:

$$\mathcal{H} = \mathcal{H}_0 - (\lambda_\perp/\pi\alpha)\mathcal{D} + \sum_{i=1,2,3} \lambda_i \mathcal{O}_i. \quad (25)$$

Where $\mathcal{H}_0$ is given by (22). For later purposes, we have suitably inverted the sign of the coupling constant $\lambda_\perp$ by making a shift of the field Φ_+: $\Phi_+ \to \Phi_+ + \sqrt{\pi/4K_+}$. In what follows, we will analyze possible phases of this model in the four sectors A, B, C, D by using a generalization of the standard variational approach [27] that accounts for ground states with nonzero values of topological charges, $\langle \partial_x \Theta_- \rangle$ and $\langle \partial_x \Phi_\pm \rangle$. The very possibility to incorporate such states within the variational method stems from the fact that the twist operators (19–21) are products of fields belonging to different Gaussian models $\mathcal{H}_0^\pm$.

3 Twistless ladder

Before addressing the role of the twist terms in (25) it is instructive first to apply the variational approach to a simpler frustrated two-leg ladder model [28,23] which, in the continuum limit, is free from parity-breaking perturbations yet being spontaneously dimerized. This model is the two leg-ladder version of the standard J_1-J_2 frustrated Heisenberg plane, in which the interchain coupling

includes besides the usual on-rung coupling, $J_\perp$, a frustrating exchange, $J_\times$, across the diagonals of the plaquettes. In the XXZ case its Hamiltonian reads:

$$H_{\text{gen}} = \sum_{a,n} \sum_{i=1,2} J_a S_i^a(n) S_i^a(n+1) + \sum_{a,n} J_a' S_1^a(n) S_2^a(n)$$
$$+ \sum_{a,n} J_a'' \left[S_1^a(n) S_2^a(n+1) + S_1^a(n+1) S_2^a(n) \right], \quad (26)$$

where J_a are defined as in (2), and

$$J_x' = J_y' = J_\perp, \quad J_z' = J_\perp \Delta',$$
$$J_x'' = J_y'' = J_\times, \quad J_z'' = J_\times \Delta'.$$

The $P_{S(L)}^{(1)} \otimes P_{S(L)}^{(2)}$ reflection symmetry of the model (26) forbids the marginal twist perturbations to appear in the continuum limit. The additional condition $J_\perp = 2J_\times$ eliminates the $n_1^a n_2^a$ part of the interchain coupling and thus makes the two decoupled Gaussian models only perturbed by the current-current interchain interaction (7) with coupling constants $g_\perp = 2J_\times \Delta' a_0$ and $g_\parallel = 2J_\times a_0$. So the bosonized continuum Hamiltonian has the structure of equation (25) with $\lambda_\perp \neq 0$ and $\lambda_{1,2,3} = 0$. As we shall see below, the dimerized phase it describes also occurs in the most part of sector C of the zigzag ladder, equation (25), where $\mathcal{D}$ is the most relevant perturbation to the Gaussian models (22).

To implement Coleman's variational procedure [27], we introduce a trial ground state,

$$|\text{vac}\rangle = |0; m_+, m_-\rangle = |0; m_+\rangle \otimes |0; m_-\rangle, \quad (27)$$

which describes free bosons in the $(\pm)$ sectors with masses $m_\pm$. These are regarded as variational parameters. To estimate the variational ground-state energy density, $E_0(m_+, m_-) = \langle \text{vac}|\mathcal{H}|\text{vac}\rangle$, one needs to normal order the Hamiltonian with the prescription that, in the normal-mode expansions of $\Phi_\pm(x)$ and $\Theta_\pm(x)$, $m_\pm$ should be treated as infrared regulator masses.

Upon normal ordering [27]

$$\mathcal{H}_0^{(\pm)} = \mathcal{N}_{m_\pm} \left[\mathcal{H}_0^{(\pm)} \right]$$
$$+ \frac{1}{8\pi v_\pm} \left[2 \left(\frac{v_\pm}{\alpha} \right)^2 + m_\pm^2 + O \left(\frac{m_\pm^2 \alpha^2}{v_\pm^2} \right) \right], \quad (28)$$

$$\cos \sqrt{4\pi K_+} \Phi_+ = \left(\frac{|m_+|\alpha}{v_+} \right)^{K_+} \mathcal{N}_{m_+} \left[\cos \sqrt{4\pi K_+} \Phi_+ \right],$$

$$\cos \sqrt{4\pi K_-} \Theta_- = \left(\frac{|m_-|\alpha}{v_-} \right)^{K_-} \mathcal{N}_{m_-} \left[\cos \sqrt{4\pi K_-} \Theta_- \right],$$

where $\mathcal{N}_{m_\pm}$ are the normal ordering symbols. Subtracting from (28) the diverging contribution of the zero-point motion when $\alpha \to 0$, ignoring for simplicity the difference between the velocities $v_\pm$ and defining the dimensionless quantities,

$$\mathcal{E} = \frac{4\pi\alpha^2}{v} E_0, \quad M_\pm = \frac{m_\pm \alpha}{v}, \quad z_\perp = \frac{4\lambda_\perp \alpha}{v}, \quad (29)$$

we find that

$$\mathcal{E}(M_+, M_-) = \frac{M_+^2 + M_-^2}{2} - z_\perp |M_+|^{K_+} |M_-|^{K_-}. \quad (30)$$

Withous loss of generality we choose $M_\pm$ to be positive. With the condition $z_\perp \ll 1$ in mind, it should be understood that the masses $M_\pm$, obtained upon minimization of $\mathcal{E}$, should satisfy $M_\pm \ll 1$.

At $d_{\mathcal{D}} > 2$ only a trivial solution exists, $M_\pm = 0$, corresponding to a critical regime in which the interchain interaction is irrelevant and the two chains asymptotically decouple in the low-energy limit. At $d_{\mathcal{D}} = K_+ + K_- < 2$ we find a nontrivial solution,

$$\frac{M_+}{\sqrt{K_+}} = \frac{M_-}{\sqrt{K_-}} = K_+^{\frac{K_+}{2(2-d_{\mathcal{D}})}} K_-^{\frac{K_-}{2(2-d_{\mathcal{D}})}} z_\perp^{\frac{1}{2-d_{\mathcal{D}}}}, \quad (31)$$

with ground state energy given by:

$$\mathcal{E}_D = - \left(\frac{1-K_+}{2K_+} \right) M_+^2 - \left(\frac{1-K_-}{2K_-} \right) M_-^2 \quad (32)$$

$$= - \left(\frac{2-d_D}{2K_+} \right) M_+^2. \quad (33)$$

This solution describes a strong-coupling, massive phase in which the fields Φ_+ and Θ_- are locked in one of infinitely degenerate minima of the potential $\mathcal{U}(\Phi_+, \Theta_-) = -(\lambda_\perp/\pi\alpha)\mathcal{D}$. Since $\lambda_\perp > 0$, these minima decouple into "even" and "odd" sets:

$$\Phi_+ = \sqrt{\frac{\pi}{4K_+}} 2n_+, \qquad \Theta_- = \sqrt{\frac{\pi}{4K_-}} 2n_-;$$
$$\Phi_+ = \sqrt{\frac{\pi}{4K_+}} (2n_+ + 1), \quad \Theta_- = \sqrt{\frac{\pi}{4K_-}} (2n_- + 1), \quad (34)$$

where $n_\pm = 0, \pm 1, \pm 2, \ldots$. The existence of these two inequivalent sets reflects two-fold degeneracy of the spontaneously dimerized ground state. Transverse dimerization is the order parameter; it is defined as $\langle \epsilon_\perp(x) \rangle$ where [7,23]

$$\epsilon_\perp(x) = \mathbf{n}_1(x) \cdot \mathbf{n}_2(x) \propto C_x^2 \cos \sqrt{4\pi K_-} \Theta_-$$
$$+ \frac{1}{2} C_z^2 \left(\cos \sqrt{4\pi K_+} \Phi_+ + \cos \sqrt{4\pi/K_-} \Phi_- \right). \quad (35)$$

Since the field Θ_- is locked, its dual Φ_- is disordered and, hence, the expectation value of the last term in (35) vanishes. Hence,

$$\langle \epsilon_\perp \rangle = \pm \epsilon_0, \quad \epsilon_0 \propto C_x^2 |M_-|^{K_-} + \frac{1}{2} C_z^2 |M_+|^{K_+}, \quad (36)$$

with the two signs of ϵ corresponding to the even and odd vacua, respectively.

The discrete (Z_2) symmetry that is spontaneously broken in the ground state is generated by even-odd interset transitions of the fields,

$$\Delta\Phi_+ = \pm\sqrt{\pi/4K_+}, \quad \Delta\Theta_- = \pm\sqrt{\pi/4K_-}, \quad (37)$$

and is related to translations by one lattice spacing on one chain only. This is not an exact symmetry of the microscopic Hamiltonian (26) but rather appears as an important property of the corresponding low-energy model with a "current-current" perturbation. The excitation spectrum of the model consists of pairs of massive topological kinks (spinons) interpolating between two adjacent minima of the potential $\mathcal{U}$. The kinks carry two topological quantum numbers – the total spin

$$S_+^z = \sqrt{\frac{K_+}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; \partial_x \Phi_+(x), \qquad (38)$$

and the relative longitudinal spin current

$$j_-^z/u = -\sqrt{\frac{K_-}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; \partial_x \Theta_-(x), \qquad (39)$$

which, according to (37), take fractional values $\pm 1/2$ [29].

4 Critical spin nematic phase

Now we are coming back to the continuum model (25) for the XXZ zig-zag ladder. We begin our discussion with sector A where the twist operator $\mathcal{O}_1$ is the only relevant perturbation to the two decoupled Gaussian models $\mathcal{H}_0^{(\pm)}$. Making a shift $\Theta_- \to \Theta_- + (1/4)\sqrt{\pi/K_-}$ we write the low-energy model in sector A as follows:

$$\mathcal{H}_A = \mathcal{H}_0^{(+)} + \mathcal{H}_0^{(-)} + \frac{2\lambda_1}{\sqrt{K_+}}\partial_x\Theta_+ \cos\sqrt{4\pi K_-}\Theta_-. \quad (40)$$

This model has the same structure as that for the XX zigzag ladder considered in reference [10]. Not surprisingly, the variational procedure we will follow now leads to qualitatively the same results as those obtained for the XX case within a symmetry-preserving mean-field approach [10].

The interaction term in (40) couples the vertex operator in the (–) channel to the topological current density $\partial_x\Theta_+$ in the (+) channel. The latter determines the z-component of the spin current which flows along the chain direction, $\hat{j}_{||}^z$,

$$\hat{j}_+^z \equiv \hat{j}_{||}^z = -v\sqrt{\frac{K_+}{\pi}}\partial_x\Theta_+. \qquad (41)$$

We observe that a finite λ_1, see equation (40), generates an additional contribution to the spin current, $\hat{j}_\perp^z$, which flows along the interchain bonds. By the continuity equation related to the conservation of the z-component of the total spin, one finds that

$$\hat{j}_\perp^z = -\frac{2}{\sqrt{\pi}}\lambda_1 \cos\sqrt{4\pi K_-}\Theta_-. \qquad (42)$$

The total spin current is therefore $\hat{j}^z = \hat{j}_{||}^z + \hat{j}_\perp^z$, and the twist operator $\mathcal{O}_1$ is nothing but a coupling term $\hat{j}_{||}^z\hat{j}_\perp^z$.

The structure of the perturbation in the model (40) suggests that the ground state admits finite values of the mass gap in the (–) channel *and* the spin current in the (+) channel. So one needs to treat both of these two quantities as variational parameters. To this end, we keep boundary conditions periodic for the field $\Theta_-(x)$ but impose twisted boundary conditions for the field $\Theta_+(x)$:

$$\Theta_+(x) = \Theta_+^0(x) - \frac{1}{v}\sqrt{\frac{\pi}{K_+}}j_+^z x. \qquad (43)$$

Here $\Theta_+^0(x)$ is a massless harmonic Bose field satisfying periodic boundary conditions, and j_+^z is the average value of the current operator (41) which is to be determined self-consistently. The variational procedure is the same as in the previous section with the exception that the ground state energy in the (+) channel will acquire a piece proportional to $(j_+^z)^2$. Otherwise this sector remains gapless: $M_+ = 0$. Using dimensionless notations,

$$\mathcal{J}_+ = \frac{2\pi\alpha}{\sqrt{K_+}v}j_+^z, \quad z_1 = 4\sqrt{\frac{\pi}{K_+}}\frac{\lambda_1\alpha}{v}, \qquad (44)$$

for the variational energy density $\mathcal{E}$ we obtain:

$$\mathcal{E}(\mathcal{J}_+, M_-) = \frac{1}{2}\left(M_-^2 + \mathcal{J}_+^2\right) \mp z_1\mathcal{J}_+ M_-^{K_-}. \qquad (45)$$

As before, we have chosen M_- to be positive. The ($\mp$) signs in the interaction term correspond to two sets of vacuum expectation values of the field Θ_-: $\Theta_- = \sqrt{\pi/K_-}n$ and $\Theta_- = \sqrt{\pi/K_-}(n + 1/2)$, respectively.

Minimizing $\mathcal{E}$ with respect to $M_\pm$ and $\mathcal{J}_+$ we find that the (–) channel is gapped,

$$M_- = K_-^{\frac{1}{2(1-K_-)}} z_1^{\frac{1}{1-K_-}}, \qquad (46)$$

if $K_- < 1$. This is actually the condition $d_1 < 2$ for the twist operator $\mathcal{O}_1$ to be a relevant perturbation, which is satisfied in sector A. At the same time, the gap supports a finite value of the spin current in the (+) channel:

$$\mathcal{J}_+ = \pm\frac{M_-}{\sqrt{K_-}} = \pm K_-^{\frac{K_-}{2(1-K_-)}} z_1^{\frac{1}{1-K_-}}. \qquad (47)$$

We notice that the dimensionless transverse current defined by

$$\hat{\mathcal{J}}_\perp = -\frac{2\pi\alpha}{\sqrt{K_+}u}\hat{j}_\perp^z = z_1 \cos\sqrt{4\pi K_-}\Theta_-, \qquad (48)$$

also acquires a finite ground-state expectation value

$$\mathcal{J}_\perp = \langle\hat{\mathcal{J}}_\perp\rangle = \mp z_1 M_-^{K_-}, \qquad (49)$$

which exactly cancels $\mathcal{J}_{||} = \mathcal{J}_+$, so that the total spin current is zero. This results in a spin nematic (or a staggered spin-flux) phase characterized by local spin currents circulating around elementary plaquettes in an alternating way. This type of ordering does not break time reversal

symmetry. In sector A the spin nematic phase is critical because the spin-density fluctuations in the (+) channel remain gapless.

We notice that the transverse current can be associated with the chirality order parameter. The latter is defined as

$$\kappa_z = \langle \kappa_z(x) \rangle, \quad \kappa_z(x) = [\mathbf{n}_1(x) \times \mathbf{n}_2(x)]_z, \qquad (50)$$

and, according to bosonization rules (16), transforms in the continuum limit to

$$\kappa_z(x) \propto \cos \sqrt{4\pi K_-} \Theta_-(x) \propto \hat{j}_\perp(x). \qquad (51)$$

As the dimerized phase discissed in Section 3, the spin nematic phase is doubly degenerate because the mixed parity symmetry $P_{S(L)}^{(1)} P_{L(S)}^{(2)}$ is spontaneously broken in the ground state. The two degenerate phases differ in the signs of the longitudinal and transverse currents. Consequently, apart from the massless bosonic mode describing low-energy fluctuations of the total magnetization, there exist massive topological Z_2 kinks corresponding to vacuum-vacuum transitions,

$$\mathcal{J}_+ \to -\mathcal{J}_+, \quad \Theta_- \to \Theta_- \pm \sqrt{\pi/4K_-},$$

and thus carrying the relative spin current $j_-^z/u = \pm 1/2$; see equation (39).

The presence of a finite longitudinal spin current in the ground state makes the transverse (xy) spin correlations incommensurate. Since the (+) channel is massless and described by a Gaussian field with a K_+ dependent compactification radius, the correlations will decay algebraically with a nonuniversal exponent. Making use of bosonization rules (16), equation (43) and the fact that the field Θ_- is locked, one easily finds the asymptotic behaviour of the transverse spin correlation function:

$$\langle S_1^+(x) S_{1,2}^- \rangle \propto \frac{(-1)^{x/a_0}}{|x|^{1/2K_+}} e^{-iq_0 x} \qquad (52)$$

where the wave vector $q_0 = \pi \mathcal{J}_\parallel / u K_+$. At the XX point ($Q = 2$, $R = 1$, $K_+ = 2$) the spin correlations decay according to the power law $|x|^{1/2}$, in agreement with reference [10].

The ground state energy of the critical spin nematic (CSN) phase is given by

$$\mathcal{E}_{\mathrm{CSN}} = -\left(\frac{1 - K_-}{2K_-}\right) M_-^2. \qquad (53)$$

Before closing this section, we would like to briefly discuss the role of the scalar operator $\cos 2\sqrt{4\pi K_-}\Theta_-$, which is generated in a higher orders and becomes relevant at $K_- < 1/2$. In the presence of a finite spin current in (+) channel, this term transforms the effective Hamiltonian in (−) channel to a double-frequency sine-Gordon model: $\mathcal{H}_0 - \lambda_{\mathrm{eff}} \sin \sqrt{4\pi K_-}\Theta_- - g \cos 2\sqrt{4\pi K_-}\Theta_-$, where $\lambda_{\mathrm{eff}} \propto \lambda_1 \langle j_+^z \rangle$. It is known [30,31] that the g-term can induce an Ising transition to a new massive phase if $g > 0$ and $g^{1/2(1-2K_-)} > \lambda_{\mathrm{eff}}^{1/(2-K_-)}$. The last inequality, however, is not satisfied since the amplitude $g \sim \lambda_{\mathrm{eff}}^2$ is rather small and, hence, the presence of the second harmonics does not qualitatively affect the above results.

5 Massive spin nematic and dimerized phases

Let us now move to sectors B and C where the properties of the systems are determined by the interplay between two most relevant perturbations, $\mathcal{O}_1$ and $\mathcal{D}$. The second twist perturbation, $\mathcal{O}_3$, is either irrelevant (as in sector B) or the least relevant (as in sector C). In Appendix A we explicitly show that its role is indeed subdominant in sectors B and C far from the SU(2)-symmetric point, $K_+ = K_- = 1$,

Thus, the effective Hamiltonian reads:

$$\begin{aligned} \mathcal{H}_{B/C} = {} & \mathcal{H}_0^{(+)} + \mathcal{H}_0^{(-)} \\ & - \frac{\lambda_\perp}{\pi\alpha} \cos \sqrt{4\pi K_+} \Phi_+ \cos \sqrt{4\pi K_-} \Theta_- \\ & + \frac{2\lambda_1}{\sqrt{K_+}} \partial_x \Theta_+ \sin \sqrt{4\pi K_-} \Theta_-. \end{aligned} \qquad (54)$$

The potential in (54) contains both the sine and cosine of the field Θ_-; so its vacuum value Θ_-^* is expected to be located somewhere within the interval $(0, \sqrt{\pi/4K_-})$ and must be such that in a massive phase with $M_- \neq 0$

$$\begin{aligned} \langle \mathrm{vac}| \cos \sqrt{4\pi K_-} \Theta_-^* |\mathrm{vac}\rangle &= M_-^{K_-}, \\ \langle \mathrm{vac}| \sin \sqrt{4\pi K_-} \Theta_-^* |\mathrm{vac}\rangle &= 0. \end{aligned} \qquad (55)$$

Setting $\Theta_- = \Theta_-^* - \gamma/\sqrt{4\pi K_-}$, we arrive at the following expression of the dimensionless variational energy:

$$\begin{aligned} \mathcal{E}(\mathcal{J}_\parallel, M_+, M_-, \gamma) = {} & \frac{1}{2}\left(M_+^2 + M_-^2 + \mathcal{J}_+^2\right) \\ & \mp z_1 \mathcal{J}_+ M_-^{K_-} \sin \gamma - z_\perp M_+^{K_+} M_-^{K_-} \cos \gamma. \end{aligned} \qquad (56)$$

Its minimization with respect to $M_\pm$, $\mathcal{J}_+$ and the angle γ yields the following set of equations:

$$M_-^{K_-}(z_\perp M_+^{K_+} \sin \gamma \mp z_1 \mathcal{J}_+ \cos \gamma) = 0, \qquad (57)$$

$$\mathcal{J}_+ \mp z_1 M_-^{K_-} \sin \gamma = 0, \qquad (58)$$

$$M_+(1 - z_\perp K_+ M_+^{K_+ - 2} M_-^{K_-} \cos \gamma) = 0, \qquad (59)$$

$$\begin{aligned} M_-(1 & - z_\perp K_- M_+^{K_+} M_-^{K_- - 2} \cos \gamma \\ & \mp z_1 K_- \mathcal{J}_+ M_-^{K_- - 2} \sin \gamma) = 0. \end{aligned} \qquad (60)$$

There are two obvious solutions of these equations in which only one of the two perturbing operators is effective. In these solutions the angle γ takes two values: 0 and $\pi/2$. The corresponding phases are, respectively: (i) a fully gapped D phase already described in Section 3, with zero current ($\mathcal{J}_+ = 0$) and nonzero masses $M_\pm$ given by equation (31), and (ii) a CSN phase with nonzero $\mathcal{J}_+$ and M_- given by equations (47, 46).

Equations (57)–(60) admit one more solution where the combined effect of the two relevant perturbations leads to an intermediate value of the mixing angle γ,

$$\cos \gamma = \frac{M_+}{M_-} \sqrt{\frac{K_-}{K_+}}, \qquad (61)$$

and a finite mass gap in the (+) channel,

$$M_+ = (z_\perp \sqrt{K_+}/z_1)^{\frac{1}{1-K_+}}. \tag{62}$$

This is a noncritical or massive spin nematic (MSN) phase characterized by the coexistence of a reduced spin current $\mathcal{J}_+$

$$\mathcal{J}_+ = [\mathcal{J}_+]_{\text{CSN}} \sin \gamma \tag{63}$$

and a nonzero dimerization

$$\epsilon_\perp \propto \pm \left[C_x^2 M_-^{K_-} \cos \gamma + \frac{1}{2} C_z^2 M_+^{K_+} \right]. \tag{64}$$

An important observation is that the minimal value of the variational energy (56) is still given by expression (32). Therefore, the energies of the MSN and CSN phases are related as

$$\mathcal{E}_{\text{MSN}} = \mathcal{E}_{\text{CSN}} + \frac{K_+ - 1}{2K_+} M_+^2.$$

So in sector B ($K_+ > 1$) the MSN phase is energetically less favorable than the CSN phase, and the ground state should be chosen between CSN and D phases. Accordingly, in sector C ($K_+ < 1$) the competing phases are MSN and D.

Consider first sector B. Here we need to compare the ground state energies of the D and CSN phases given by equations (33, 53). These are of the same order when the mass gaps of the two phases, equations (31, 46), become comparable. Notice that the coupling constants z_1 and $z_\perp$ are both proportional to $g_\perp$ and, hence are of the same order of magnitude; their ratio is

$$z_\perp/z_1 = C\sqrt{K_+}, \tag{65}$$

where C is a nonuniversal number. Therefore, the condition

$$z_\perp^{\frac{2}{2-d_D}} \sim z_1^{\frac{2}{2-d_1}} \tag{66}$$

can be satisfied only in some vicinity of the line $K_+ = 1$ where scaling dimensions of the operators $\mathcal{D}$ and $\mathcal{O}_1$ become equal.

As already mentioned, it is not possible to establish a precise relationship between the parameters of the original, microscopic model (3) and the effective low-energy theory (25). As a result, the parameter C in (65) is unknown. Therefore we are forced to consider two cases, $z_1 > z_\perp$ and $z_1 < z_\perp$, on equal footing and draw plausible scenarios for each of them, leaving the final choice to future numerical work.

Setting $K_+ = 1 + \delta$ with $|\delta| \ll 1$, we find that the condition (66) translates to the relation

$$\delta = (1 - K_-)\frac{\ln(z_\perp/z_1)}{\ln(1/z_1)}. \tag{67}$$

This relation determines a line $\delta = \delta(K_-)$ which lies entirely in sector B ($\delta > 0$) and is located very close to the line $K_+ = 1$ only if $z_1 < z_\perp$. Under this condition the relation (67) determines a phase boundary between the CSN and D states. The transition is of first order, associated
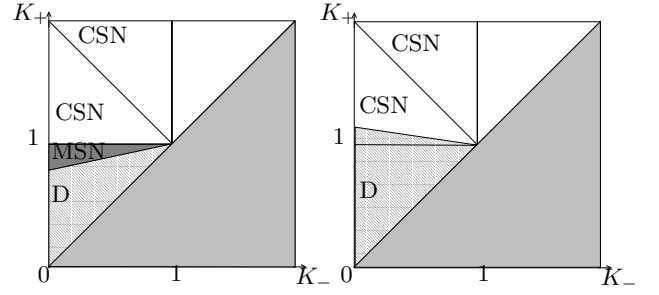

Fig. 2. Possible phase diagrams. The left panel corresponds to the case $z_\perp < z_1$ when a narrow region occupied by the massive spin nematic (MSN) phase intrudes between the dimerized (D) and critical spin nematic (CSN) phases. The right panel is the case $z_\perp > z_1$ where the D and CSN phases are separated by a first-order.

with discontinuities of the spin current and dimerization order parameters. It can be easily shown that in the case $z_1 < z_\perp$ the D phase, occupying a narrow region close to the line $K_+ = 1$, extends over the whole C phase.

In the opposite case, $z_1 \geq z_\perp$, equation (67) has no solution for $\delta > 0$, implying that CSN is a stable ground state in the whole sector B. Moving to sector C opens a possibility for the MSN phase. If $1 - K_+ \ll 1$, the condition $z_1 \geq z_\perp$ admits a small nonzero mass M_+ given by (62). Thus is sector C the upper boundary for the MSN phase is $K_+ = 1$. The lower boundary is found from the requirement $\cos \gamma < 1$ (see Eq. (61)). Within the logarithmic accuracy, this brings us again to equation (67), this time for $\delta(K_-) < 0$.

Thus, if the ratio $z_1/z_\perp > 1$, then the CSN and D phases are "sandwiched" by the MSN phase occupying a narrow region in sector C

$$1 - (1 - K_-)\frac{\ln(z_1/z_\perp)}{\ln(1/z_1)} < K_+ < 1 \tag{68}$$

attached to the line K_+ (see Fig. 2). In all this region $\mathcal{E}_{\text{MSN}} < \mathcal{E}_{\text{D}}$. The transitions that occur on the upper and lower boundaries of the MSN phase are continuous. When moving from sector B to sector C through the MSN phase, the mixing angle γ varies from $\pi/2$ to 0. Correspondingly, the current $\mathcal{J}_+$ decreases from its nominal value $[\mathcal{J}_+]_{\text{CSN}}$ and vanishes at the lower boundary, whereas the transverse dimerization $\epsilon_\perp$ increases from zero at the upper boundary and reaches its value ϵ_0, equation (36), in the pure D phase at the lower boundary (see Fig. 2). A possible way of driving the ladder to pass through these phases is shown in Figure 3.

6 Ferromagnetic phase

Let us now consider sector D where $K_+ > K_- > 1$. This condition implies that $K_+ K_- = R^2 > 1$, and so this sector corresponds to the case of a ferromagnetic interchain interaction ($g_\parallel < 0$). The effective low-energy model

$$\mathcal{H}_D = \mathcal{H}_0^{(+)} + \mathcal{H}_0^{(-)} + \lambda_2 \sqrt{K_+} \partial_x \Phi_+ \sin \sqrt{4\pi/K_-}\Phi_- \tag{69}$$

contains only one relevant twist operator and is dual to model (40) describing the SN phase in sector A: mapping between these two models is achieved by duality transformations

$$2\lambda_1 \to \lambda_2, \quad K_\pm \to 1/K_\pm, \quad \Phi_\pm \to \Theta_\pm. \qquad (70)$$

Using this correspondence, we can readily translate the results of Section 4 to the present case. In particular, the spontaneously generated spin current $\mathcal{J}_+$ of the SN phase transforms to the z-component of the uniform spin density. So the ground state of the system in sector D is *ferromagnetic* (F). Contrary to the spin nematic phase, the F phase breaks time reversal invariance but preserves parity $P_S^{(1)} \otimes P_L^{(2)}$.

Shifting the field Φ_- by $\sqrt{\pi K_-}/4$ and passing to dimensionless notations for the coupling constant,

$$z_2 = 2\sqrt{\pi K_+}\frac{\lambda_2 \alpha}{v}$$

and total magnetization

$$\mathcal{S}_+ = \frac{2\pi\alpha}{\sqrt{K_+}}m^z,$$

we write the variational energy density as

$$\mathcal{E} = \frac{1}{2}\left(M_-^2 + \mathcal{S}_+^2\right) \mp z_2\mathcal{S}_+ M_-^{1/K_-}. \qquad (71)$$

Its minimization yields a finite gap in the $(-)$ channel,

$$M_- = K_-^{-\frac{K_-}{2(K_- -1)}} z_2^{\frac{K_-}{K_- -1}} \qquad (72)$$

which supports a nonzero magnetization directed along the exchange anisotropy axis:

$$\mathcal{S}_+ = \pm\sqrt{K_-}M_- = \pm K_-^{-\frac{1}{2(K_- -1)}} z_2^{\frac{K_-}{K_- -1}}. \qquad (73)$$

The $(+)$ channel remains gapless. Together with the finite spontaneous magnetization this circumstance makes the longitudinal spin correlations algebraic *and* incommensurate:

$$\langle S_1^z(x) S_{1,2}^z(0)\rangle = \langle S^z\rangle^2 - \frac{K_+}{8\pi^2}\frac{1}{x^2} + \text{const}\frac{(-1)^{x/a_0}}{|x|^{K_+/2}}\cos q_0 x, \qquad (74)$$

where $q_0 = \mathcal{S}_+\sqrt{K_+}/2\alpha$. The transverse spin correlations are short-ranged.

The ground state is doubly degenerate: the two vacua transforming to each other under time reversal. The corresponding topological kinks have a finite mass gap and carry the relative spin

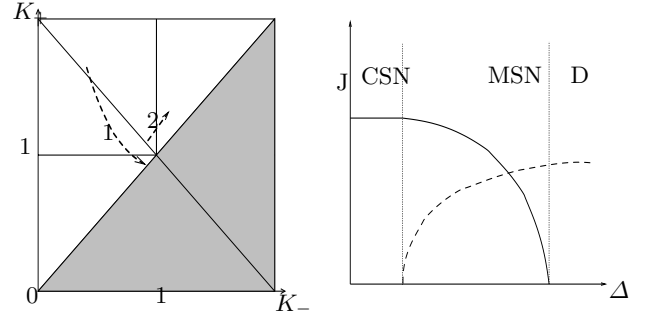$$S_-^z = \frac{1}{\sqrt{\pi K_-}}\int_{-\infty}^{\infty} \mathrm{d}x\ \partial_x \Phi_- = \pm\frac{1}{2}.$$

Fig. 3. Left panel: two possible paths in the phase space of the model parameters. Along the path "1" Δ is increased with Δ' kept constant at $z_\perp < z_1$. In this case the model undergoes two transitions, first from the critical spin nematic (CSN) phase to a massive spin nematic (MSN) phase, and then from the MSN phase to the dimerized (D) phase. The right panel shows the qualitative behavior of the spin current (solid line) and the dimerization order parameter (dashed line) along this path. The path labeled by "2" corresponds to decreasing Δ' with Δ constant. Along this path the model moves from the CSN phase towards a ferromagnetic one.

7 RG approach at A-D boundary

The results of Sections 4 and 6 are valid far enough from the boundary between sectors A and D where one of the two twist operators, $\mathcal{O}_{1,2}$, is strongly relevant while the other is strongly irrelevant. On the boundary $K_- = 1$ separating these sectors both twist operators become marginal. Therefore we can expect that in the immediate vicinity of the boundary,

$$K_- = 1 - \delta_-, \quad |\delta_-| \ll 1, \qquad (75)$$

far away from the SU(2)-symmetric point, i.e. $K_+ > 1$ in the sense that $K_+ - 1 = O(1)$, the infrared behavior of model will be controlled by the interplay between the two parity-breaking operators with a nonzero conformal spin, $\mathcal{O}_{1,2}$, and the longitudinal (conformal-scalar) terms $\partial_x \Phi_R^\pm \partial_x \Phi_L^\pm$ responsible for renormalization of the coupling constants. So the starting low-energy model should therefore contain both twist terms:

$$\mathcal{H} = \mathcal{H}_0^{(+)} + \mathcal{H}_0^{(-)} + \gamma_1 \partial_x \Theta_+ \sin\tilde{\beta}\Theta_- + \gamma_2 \partial_x \Phi_+ \sin\beta\Phi_-. \qquad (76)$$

Here $\gamma_{1,2}$ differ from $\lambda_{1,2}$ by some multiplicative factors, and

$$\tilde{\beta} = \frac{4\pi}{\beta} = \sqrt{4\pi K_-} = \sqrt{4\pi}\left[1 + \frac{\delta_-}{2} + O(\delta_-^2)\right]. \qquad (77)$$

Notice that the two twist terms in (76) contain vertex operators (the sines) of mutually dual and *nonlocal* fields, Θ_- and Φ_-. Models of this kind cannot be treated by the variational method used in the preceding sections. This is why in this section we address the RG flow of this model which will be studied using a mapping of the bosonic Hamiltonian (76) onto a theory of four interacting real (Majorana) fermions (cf. Ref. [10]).

Let us first make all perturbations in (76) strictly marginal. This can be done by the following rescaling of the fields in the (−) and (+) sectors:

$$\Phi_- \to \sqrt{K_-}\Phi_-, \quad \Theta_- \to (1/\sqrt{K_-})\,\Theta_- \qquad (78)$$
$$\Phi_+ \to \sqrt{q_+}\Phi_+, \quad \Theta_+ \to (1/\sqrt{q_+})\,\Theta_+ \qquad (79)$$

The meaning of the first rescaling is transparent: we enforce the twist operators in (76) to have the scaling dimension 2. This rescaling generates a current-current term in the (−) channel. On the other hand, fusing the two twist operators, one generates a similar term in the (+) sector; that will renormalize the parameter K_+, $K_+ \to \bar{K}_+ = K_+ q_+$. Below we will set $q_+ = 1 - \delta_+$ assuming that $|\delta_+| \ll 1$. The Hamiltonian (76) then acquires the form:

$$\mathcal{H} = \frac{u}{2}\sum_{s=\pm}\left[(\partial_x\Phi_s)^2 + (\partial_x\Theta_s)^2\right] - 2u\sum_{s=\pm}\delta_s\partial_x\Phi_{sR}\partial_x\Phi_{sL}$$
$$+ \gamma_1\partial_x\Theta_+ \sin\sqrt{4\pi}\Theta_- + \gamma_2\partial_x\Phi_+ \sin\sqrt{4\pi}\Phi_- \qquad (80)$$

where $\delta_\pm$ satisfy the initial conditions

$$\delta_+^{(0)} = 0, \quad \delta_-^{(0)} = \delta_-. \qquad (81)$$

It is understood that extra factors appearing due to rescaling of K_+ are absorbed into a redefinition of the coupling constants γ_1 and γ_2.

The structure of (80) immediately suggests mapping onto four real (Majorana) fermions, ξ^a $(a = 0,1,2,3)$. This can be done using a correspondence

$$(\Phi_+, \Theta_+) \Rightarrow (\xi_1, \xi_2), \quad (\Phi_-, \Theta_-) \Rightarrow (\xi_3, \xi_0)$$

and standard fermionization rules for the currents and vertex operators. The resulting theory is given by the Euclidean action describing four degenerate massless fermions with a *chirally asymmetric* interaction:

$$S = \sum_{a=0}^{3}\int d^2z\,(\xi^a\bar{\partial}\xi^a + \bar{\xi}^a\partial\bar{\xi}^a)$$
$$+ 2\pi v\int d^2z\,[\delta_+\,\xi_1\xi_2\bar{\xi}_1\bar{\xi}_2 + \delta_-\,\xi_3\xi_0\bar{\xi}_3\bar{\xi}_0$$
$$+ \gamma_+(\xi_1\xi_2\xi_3\bar{\xi}_0 + \bar{\xi}_1\bar{\xi}_2\bar{\xi}_3\xi_0)$$
$$+ \gamma_-(\xi_1\xi_2\xi_0\bar{\xi}_3 + \bar{\xi}_1\bar{\xi}_2\bar{\xi}_0\xi_3)]. \qquad (82)$$

Here $\xi_a(z)$ and $\bar{\xi}_a(\bar{z})$ are holomorphic (left) and antiholomorphic (right) components of the Majorana fields, $z = v\tau + ix$ and $\bar{z} = v\tau - ix$ are complex coordinates, $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$, and

$$\gamma_\pm = \frac{\pi^{3/2}\alpha}{2\pi u}(\gamma_1 \pm \gamma_2). \qquad (83)$$

Due to its chiral asymmetry, the interaction in (82) gives rise to renormalization of the velocities already on the one-loop level. For this reason we will discriminate between the velocities of different Majorana species, and set $v_a = v(1 - 4\pi\rho_a)$ $(a = 0,1,2,3)$, where the dimensionless parameters

ρ_a are subject to renormalization with initial conditions $\rho_a^{(0)} = 0$.

Using the standard fusion rules for fermion fields [33], one can easily derive the following one-loop RG equations:

$$\dot{\delta_+} = -2\gamma_+\gamma_-, \quad \dot{\delta_-} = 0, \qquad (84)$$
$$\dot{\gamma_+} = \delta_-\gamma_-, \quad \dot{\gamma_-} = \delta_-\gamma_+, \qquad (85)$$
$$\dot{\rho_1} = \dot{\rho_2} = \gamma_+^2 + \gamma_-^2, \qquad (86)$$
$$\dot{\rho_3} = \gamma_+^2, \quad \dot{\rho_0} = \gamma_-^2, \qquad (87)$$

where $\dot{g} \equiv dg(l)/dl$, $l = \ln(L/\alpha)$.

First of all, we observe that the coupling constant δ_- stays unrenormalized:

$$\delta_-(l) = \delta_-(0) = \delta_-. \qquad (88)$$

Representing $\lambda_\pm$ as

$$\gamma_\pm = g_1 \pm g_2, \quad g_{1,2} = (\pi^{3/2}\alpha/2\pi u)\gamma_{1,2}$$

we rewrite the first, third and fourth RG equations as

$$\dot{\delta_+} = -2\left(g_1^2 - g_2^2\right) \qquad (89)$$
$$\dot{g_1} = \delta_-g_1, \quad \dot{g_2} = -\delta_-g_2 \qquad (90)$$

We see that, depending on the sign of δ_-, either $g_1(l)$ or $g_2(l)$ grow up upon renormalization:

(a) $\underline{\delta_- > 0}$

$$g_1(l) = g_1^{(0)}e^{\delta_-l}, \quad g_2(l) = g_2^{(0)}e^{-\delta_-l} \to 0. \qquad (91)$$

Strong-coupling behavior of $g_1(l)$ in (91) is associated with a dynamical generation of a mass gap

$$m_1 \propto |g_1|^{1/\delta_-}. \qquad (92)$$

(b) $\underline{\delta_- < 0}$

$$g_2(l) = g_2^{(0)}e^{|\delta_-|l}, \quad g_1(l) = g_1^{(0)}e^{-|\delta_-|l} \to 0. \qquad (93)$$

Here the mass gap is estimated as

$$m_2 \propto |g_2|^{1/|\delta_-|}. \qquad (94)$$

The cases (a) and (b) describe the CSN and F phases, respectively. Estimations (92), (94) are consistent with the power-law scaling of the corresponding mass gaps, equations (46, 72). In both cases, $|\delta_+(l)|$ flows to strong coupling. It goes to large negative values in the case (a), implying that K_+ becomes even larger upon renormalization. In the case (b) it flows to large positive values; so the effective K_+ significantly reduces, and that might indicate the importance of the neglected twist operator $\mathcal{O}_3$. Stability of the F phase is therefore under question.

Exactly at the boundary between sectors A and D $\delta_- = 0$. In this case both g_1 and g_1 stay unrenormalized. Moreover,

$$\delta_+(l) = -2(g_1^2 - g_2^2)l. \qquad (95)$$

So, if in addition we set $g_1 = g_2$, the 1-loop RG will display a weak-coupling regime for all coupling constants. This is the self-dual point of the model where the interaction is not renormalized: for all effective couplings parametrizing interaction the β–function vanishes. Amazingly, in this case the Majorana action (82) decouples into two chirally asymmetric, independent parts, $S = S_{\text{I}} + S_{\text{II}}$, where

$$S_{\text{I}} = \int \mathrm{d}^2 z \ \left(\sum_{a=1}^{3} \xi^a \bar{\partial} \xi^a + \bar{\xi}^0 \partial \bar{\xi}^0 + g \xi^1 \xi^2 \xi^3 \bar{\xi}^0 \right) \quad (96)$$

and S_{II} is obtained from S_{I} by reversing the chiralities of all the fields. Notice that even though the present case corresponds to an essentially anisotropic regime (remember that we are far away from the SU(2)-symmetric point of the model), the effective theory on the boundary between sectors A and D ($K_- = 1$) with the self-duality condition $g_1 = g_2$ exhibits an enlarged, chiral SO(3) $\otimes$ SO(3) symmetry. Consistent with this symmetry is renormalization of the velocities. The velocity of the singlet fermion, $\bar{\xi}^0$, stays intact: $\dot{\rho}_0 = 0$. However, the triplet velocity is renormalized. The RG equation $\dot{\rho}_t = 4g^2$ ($\rho_i \equiv \rho_t$ $i = 1, 2, 3$) shows that $4\pi\rho_t(l)$ increases upon renormalization and reaches values of the order of 1 in the region where $g^2 l \sim 1$. This sets up an infrared energy scale in the problem, $\omega_0 \sim \Lambda \exp(-\mathrm{const}/g^2)$, at which the triplet collective excitations soften significantly. This is in agreement with the exact results for the spectrum of model (96), recently obtained by Tsvelik [32].

Interestingly enough, the exact solution [32] shows that the chiral SO(3) symmetry of the action (96) is spontaneously broken at $T = 0$, and the ground state of the model represents a "chiral ferromagnet" characterized by a nonzero expectation value of the vector current:

$$\langle \mathbf{I} \rangle \neq 0, \quad I^a = -(i/2)\epsilon^{abc}\xi_b\xi_c.$$

Similarly, for action S_{II}

$$\langle \bar{\mathbf{I}} \rangle \neq 0, \quad \bar{I}^a = -(i/2)\epsilon^{abc}\bar{\xi}_b\bar{\xi}_c.$$

As long as the actions S_{I} and S_{II} are decoupled, there is no correlation between $\langle \mathbf{I} \rangle$ and $\langle \bar{\mathbf{I}} \rangle$, or equivalently, between the magnetization $\mathbf{m} = \langle \mathbf{I} \rangle + \langle \bar{\mathbf{I}} \rangle$ and spin current $\mathbf{j} = \langle \mathbf{I} \rangle - \langle \bar{\mathbf{I}} \rangle$. Such correlation appears upon deviation from the A–D boundary since in this case chirally-symmetric terms that couple the actions S_{I} and S_{II} (and also introduce a finite XXZ anisotropy) are generated. Thus, in the A-vicinity of the A–D boundary $m_z \neq 0$, $j_z = 0$, whereas in the D-vicinity the situation is just inverted: $m_z = 0$, $j_z \neq 0$. So, the resulting picture at the A–D boundary depends on the side from which this boundary is approached, implying that the CSN – F transition is first-order.

Even though the action $S = S_{\text{I}} + S_{\text{II}}$ provides the simplest field-theoretical model for a frustrated ladder with a chirally asymmetric interaction and, hence, is quite interesting in its own right, we will refrain from its further discussion because it does not account for the low-energy properties of the zigzag spin-1/2 ladder with the *generic* SU(2) symmetry (the point $K_+ = K_- = 1$).

8 Conclusions

In this paper we have analytically derived the phase diagram of the spin-1/2 anisotropic zigzag ladder with a weak interchain coupling ($J' \ll J$). Using the Abelian bosonization method combined with a variational approach, we have found that, depending on the anisotropy parameters, the system may occur in several possible phases. These include spontaneously dimerized phases which may or may not preserve parity, phase D and massive spin nematic (MSN), respectively. In addition, there can exist a non-dimerized, parity-breakin phase, which is the critical spin nematic phase (CSN), as well as a ferromagnetic phase (F). While some of these phases have already been anticipated earlier (see Refs. [10–14]), the detailed analysis of the MSN phase, as well as the duality between the F and CSN phases, discussed above, represent novel results which provide a deeper characterization of the whole phase diagram of the model. In particular, we have shown that the CSN phase extends well beyond the XX limit and covers broad regions in the parameter space of the XXZ model (see sectors A and B in Fig. 1). We have also shown that each of these phases is characterized by topological excitations carrying fractional quantum numbers.

Starting from a vicinity of the XX point, we addressed the nature of the transition between the CSN and D phases taking place upon increasing the intra-chain anisotropy parameter Δ, say, at a fixed positive value of Δ'. As follows from our variational analysis, this transition can occur either directly being discontinuous, or in two steps, i.e. via a formation of an intermediate MSN phase (see Fig. 2). Which of these two possible scenarios can be realized depends on fine details of the microscopic model. The control parameter of the low-energy effective theory is the ratio $z_1/z_\perp$ between the amplitudes of the main competing perturbations — the twist and dimerization operators. If $z_\perp > z_1$, the CSN – D transition is first order. In the opposite case, $z_\perp < z_1$, the CSN and D phases are sandwiched by the MSN phase characterized by the coexistence of a finite spin current with dimerization. The variational approach suggests that CSN–MSN and MSN – D transitions are continuous. Even though this approach is known to be inadequate to identify universality classes of the transitions, we would argue that the CSN–MSN transition is of the Berezinskii-Kosterlitz-Thouless type, while the MSN – D is of the Ising type.

As we already emphasized, the effective low-energy model which has been studied throughout this paper actually represents a wide class of frustrated spin chains sharing the same symmetry properties with the model (1). Hence we cannot a priori predict which of the above scenarios would occur for the Hamiltonian (1). The accuracy of the recent DMRG calculations [11,12] for the $S = 1/2$ zigzag ladder (3) was reported to be inadequate to resolve this issue. Moreover, the non-linear σ model approach of reference [17], which excludes the massive chiral phase for half integer spins, is only valid in the vicinity of the classical Liftshitz point $j = 1/4$.

Appendix A: More about sector C

In this appendix we address the role of the so far neglected twist perturbation $\lambda_3 \mathcal{O}_3$ which becomes relevant in sector C. Adding this term to the effective Hamiltonian leads us to the following theory:

$$\mathcal{H}_C = \mathcal{H}_0^{(+)} + \mathcal{H}_0^{(-)}$$
$$- \frac{\lambda_\perp}{\pi\alpha} \cos\sqrt{4\pi K_+}\Phi_+ \cos\sqrt{4\pi K_-}\Theta_-$$
$$+ \frac{2\lambda_1}{\sqrt{K_+}} \partial_x\Theta_+ \sin\sqrt{4\pi K_-}\Theta_-$$
$$+ \frac{\lambda_3}{\sqrt{K_-}} \partial_x\Phi_- \sin\sqrt{4\pi K_+}\Phi_+. \tag{A.1}$$

It is convenient to introduce dimensionless notations for the coupling constant

$$z_3 = 2\sqrt{\frac{\pi}{K_-}}\frac{\lambda_3\alpha}{v}$$

and the z-component of the relative spin density,

$$\mathcal{Q}_- = 2\sqrt{\pi}\alpha\partial_x\Phi_-.$$

The variational approach we followed in Section 5 is straightforwardly generalized for the present case. As compared to Section 5, here we have two additional variational parameters: the relative spin density $\mathcal{Q}_-$ and a mixing angle ζ for the field Φ_+. The variational energy then depends on six variables:

$$\mathcal{E}_C = \frac{1}{2}(M_+^2 + M_-^2 + \mathcal{J}_+^2 + \mathcal{Q}_-^2)$$
$$\mp z_1\mathcal{J}_+ M_-^{K_-}\sin\gamma \mp z_3\mathcal{Q}M_+^{K_+}\sin\zeta$$
$$- z_\perp M_+^{K_+} M_-^{K_-}\cos\gamma\cos\zeta. \tag{A.2}$$

Its minimization yields the following set of equations:

$$M_-^{K_-}(z_\perp M_+^{K_+}\sin\gamma\cos\zeta \mp z_1\mathcal{J}_+\cos\gamma) = 0,$$
$$M_+^{K_+}(z_\perp M_-^{K_-}\cos\gamma\sin\zeta \mp z_3\mathcal{Q}_-\cos\zeta) = 0,$$
$$\mathcal{J}_+ \mp z_1 M_-^{K_-}\sin\gamma = 0,$$
$$\mathcal{Q}_- \mp z_3 M_+^{K_+}\sin\zeta = 0,$$
$$M_-(1 \mp z_1\mathcal{J}_+ K_- M_-^{K_- - 2}\sin\gamma$$
$$- z_\perp K_- M_+^{K_+} M_-^{K_- - 2}\cos\gamma\cos\zeta) = 0,$$
$$M_+(1 \mp z_3\mathcal{Q}_- K_+ M_+^{K_+ - 2}\sin\zeta$$
$$- z_\perp K_+ M_-^{K_-} M_+^{K_+ - 2}\cos\gamma\cos\zeta) = 0. \tag{A.3}$$

There exist solutions of these equations in which the second twist perturbation $\lambda_3 \mathcal{O}_3$ plays no role:

(i) $\gamma = \zeta = 0$, $\mathcal{J}_+ = \mathcal{Q}_- = 0$ – D phase;
(ii) $\gamma = \pi/2$, $\zeta = 0$, $\mathcal{Q}_- = 0$ – CSN phase;
(iii) $0 < \gamma < \pi/2$, $\zeta = 0$, $\mathcal{Q}_- = 0$ – MSN phase.

There exists a pair of solutions which are "dual" to (ii) and (iii), i.e. can be obtained from the latter by the replacements $z_1 \to z_3$, $\mathcal{J}_+ \to \mathcal{Q}_-$, $M_+ \leftrightarrow M_-$:

(iv) $\gamma = 0$, $\zeta = \pi/2$, $\mathcal{J}_+ = 0$. This is a critical phase with a nonzero relative magnetization (CRM), $\mathcal{Q}_- \neq 0$;
(v) $\gamma = 0$, $0 < \zeta < \pi/2$, $\mathcal{J}_+ = 0$ – the massive version of the above phase (MRM). In these two phases the twist operator $\mathcal{O}_1$ plays no role.

There also exist solutions in which both twist perturbations are effective. One of them corresponds to the case

(vi) $\gamma = \pi/2$, $\zeta = \pi/2$, in which the $z_\perp$-perturbation is ineffective and the variational energy decouples into a direct sum $\mathcal{E}_{CSN} + \mathcal{E}_{CRM}$. The resulting phase is fully gapped and represents a mixture of CSN and CRM phases – mixed (M) phase with nonzero $\mathcal{J}_+$ and $\mathcal{Q}_-$.

The case of arbitrary values of the mixing angles, $\gamma, \zeta \neq 0, \pi/2$, should be abandoned because, as follows from equations (A.3), it requires that $z_1^2 z_3^2 = z_\perp^2$, — a condition which represents just a point in the parameter space of the model and which, on the other hand, cannot be satisfied for all coupling constants being of the same order.

The minimal value of the variational energy is again given by equation (32). From this expression it is obvious that in sector C ($K_\pm < 1$) the M phase has a lower energy than each of its "constituents", i.e. CSN and CRM phases. So we are left to find out if the M and MRM phases can compete with the D and MSN phase.

Consider the MRM phase assuming the most favorable condition $z_3 > z_\perp$. Since the MRM phase is "dual" to the MSN phase, from equation (68) we can read off the range where it can exist:

$$1 - (1 - K_+)\frac{\ln(z_3/z_\perp)}{\ln(1/z_3)} < K_- < 1. \tag{A.4}$$

We see that, except for an extremely unrealistic case $z_3^2 > z_\perp$, the condition (A.4) determines a vicinity of the negative semi-axis $K_- = 1$, $K_+ < 1$, which is located in the unphysical part of the (K_+, K_-) plane, well beyond sector C. Thus the MRM phase should be abandoned.

Let us compare the energies of the M and MSN phases. In both cases the mass M_- is given by the same expression, so we only need to compare the masses in the $(+)$ channel. Comparing the mass M_+ in the M phase, $M_+ \sim z_3^{1/(1-K_+)}$, with that in the MSN phase, equation (62), we find that, except for extremely small values of $z_\perp$, namely $z_\perp < z_1 z_3$, the MSN phase is always more favorable.

Finally, we are left to compare the energies of the M and D phases. On one hand, in the D phase we are below

the line (68). This means that $z_\perp^{\frac{1}{2-k_D}} > z_1^{\frac{1}{1-K_-}}$ implying that the mass gap of the D phase is greater than the mass M_- of the M phase. On the other hand, to the left of the MSD-D transition line (A.4) we have the condition $z_\perp^{\frac{1}{2-k_D}} > z_3^{\frac{1}{1-K_+}}$ that tells us that the mass of the D phase is greater than the mass M_+ of the M phase. Consequently, the D phase is energetically more favorable than the M phase.

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